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Nonlinear analysis of a differential–difference equation with next-nearest-neighbour interaction for traffic flow

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Abstract

We analyse the Newell–Whitham-type car-following model described by a differential–difference equation with a generalized optimal velocity function which depends not only on the headway of each car but also on the headway of the immediately preceding one. Linear stability analysis shows that the model is stabilized for larger delay time by taking into account the headway of the immediately preceding car. From the nonlinear analysis, the propagating kink solution for traffic jams is obtained. The fundamental diagram and the relation between the headway and the delay time are examined by numerical simulation. We find that the result from the nonlinear analysis is in good agreement with that obtained from the numerical simulation.

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1. Introduction

Traffic flow problems have been intensively studied on the basis of fluid-dynamical models [1–6], cellular automaton models [7–14], and car-following models [15–19]. Many traffic models have been proposed and the mathematical features of the models have been revealed [20]. The advantage of the car-following models is that one can examine the analytical structure in the models. One car-following model was proposed and analysed by Newell [15] and Whitham [16]. The model is given by a first-order differential–difference equation by introducing a delay time which plays an important role in the occurrence of traffic congestion. Another extended version of a car-following model was proposed by Bando *et al* [17–19] without introducing a delay time. This model is described by a second-order differential equation with an optimal velocity function. The spontaneous transitions from freely moving traffic to congested traffic have been clarified by linear stability analysis and numerical simulation. In the congested flow, the uniform flow becomes unstable and a stop-and-go state appears. These phenomena have been found experimentally from data on highway traffic [21–25]. Furthermore, nonlinear analysis of an optimal velocity model has been performed by Komatsu and Sasa [26]. They have applied a modified Korteweg–de Vries (MKdV) equation to a traffic jam which is described by a kink–antikink density wave. The Newell–Whitham-type car-following model described by an optimal velocity function has also been analysed in detail [27–31]. The exact solutions to the first-order differential–difference equations with the optimal velocity function have been found.

Many approaches to extending the model toward a realistic traffic model have been pursued [32–42]. The coupled-map model [32–35] was proposed for investigating open systems such as junctions and ramps, and to move toward two- or multi-lane models. To obtain a realistic value of acceleration, a generalized force model [36] was proposed and asymmetry of sensitivity between the acceleration and deceleration processes [37] was introduced. Another approach to generalizing the optimal velocity function is to take into account backward reference [38] and car interaction before the car ahead [39]. The optimal velocity model which includes car interaction before the car ahead is examined in the context of a difference equation [39] and second-order differential equation [41, 42]. To the best of our knowledge, a first-order differential–difference equation has not yet been examined within the generalized optimal velocity model.

The purpose of this paper is to analyse the Newell–Whitham-type car-following model described by differential–difference equations with an optimal velocity function which depends not only on the headway of each car but also on the headway of the immediately preceding one. Linear stability analysis shows that the model is stabilized for larger delay time by taking into account the headway of the immediately preceding car. From the nonlinear analysis, it is shown that the traffic congestion is described by the MKdV equation. We will compare the analytical result with that from the numerical simulation.

This paper is organized as follows. In section 2 the generalized optimal velocity model described by a differential–difference equation is presented, taking into account the next-nearest-neighbour interaction. In section 3 the kink solution for traffic jams is obtained from the nonlinear analysis. In section 4 the numerical simulation is carried out by examining the relation between the density and the flow, and the relation between the headway and the delay time. The simulation result is compared with the analytical result. Section 5 is devoted to the conclusions.

2. Model

We will consider a traffic model given by differential–difference equations of the form

$$\dot{x}_n(t+\tau) = (1-p)V(\Delta x_n) + pV(\Delta x_{n+1}) \tag{1}$$

where $x_n(t)$ is the position of the nth car at time t, $\Delta x_n(t) = x_{n+1}(t) - x_n(t)$ represents the headway of the car, and hence $\Delta x_{n+1}(t)$ is the headway of the immediately preceding car. The dot on the left-hand side of equation (1) represents the derivative with respect to time t, and τ is the delay time which is the time lag before reaching optimal velocity. In equation (1), $n = 1, 2, \ldots, N$ represents each car number, with N being the total number of cars. In this paper, we will consider a periodic boundary condition with respect to the coordinate x_n with period L. According to the original optimal velocity function proposed by Bando *et al* [17], we will take a hyperbolic tangent function of the form

$$V(\Delta x_n) = \tanh(\Delta x_n - b_c) + \tanh(b_c) \tag{2}$$

with $b_c = 2$ which gives the safe distance. In order to deal with a more realistic traffic model, we have to choose the parameters τ, b_c and the value of the maximum velocity so as to fit to experimental data such as the flux–density relation. However, we will not consider this here, because we are interested in the characteristic properties of our model.

The value of the parameter p in equation (1) reflects the effect of taking into account the car interaction before the car ahead. The driver sometimes pays attention to not only the headway but also the headway of the immediately preceding car. If the headway of the preceding car is small, the driver assumes that the forward driver is decelerating; thus the driver decreases the optimal velocity even though the headway of his/her car is long enough. On the other hand, if the headway of the preceding car is long, the driver assumes that the forward driver is accelerating; thus the driver increases the optimal velocity even though the headway of his/her car is short. Now, the generalized optimal velocity function given by the right-hand side of equation (1) has these properties. For larger values of p, the effect of taking into account the headway of the immediately preceding car becomes large.

Assuming that the delay time τ is small, Taylor expansion in equation (1) leads to

$$\ddot{x}_n(t) = a\left((1-p)V(\Delta x_n) + pV(\Delta x_{n+1}) - \dot{x}_n(t)\right) \tag{3}$$

where we have put $a = 1/\tau$, which is called the sensitivity. This model has been analysed in [41,42]. The purpose of this paper is to examine the model given by equation (1) analytically and numerically.

The model given by equation (1) has a uniform-flow solution $x_n^{(0)}(t) = bn + ct$ with $b = L/N$ and $c = V(b)$. Examining the stability against a small perturbation around the uniform-flow solution, we find that the stability condition is given by

$$\tau < \frac{1+2p}{2V'(b)}. \tag{4}$$

Therefore, by taking into account the headway of the immediately preceding car, the model is stabilized for larger delay time compared with that in the case of $p = 0$.

3. Nonlinear analysis

We examine the generalized optimal velocity model by the nonlinear analysis method. In the long-wavelength approximation, we can find the dispersion relation with respect to k near the critical point:

$$z = V'(b_c)\mathrm{i}k + V'(b_c)(V'(b_c)\tau - \tfrac{1}{2}(1+2p))k^2 - V'(b_c)\tfrac{1}{24}(1+12p-12p^2)\mathrm{i}k^3 - V'(b_c)\tfrac{1}{48}(1+6p+36p^2-40p^3)k^4 + \mathrm{O}(k^5). \tag{5}$$

The terms proportional to k^3 and k^4 in equation (5) are the dispersion term and dissipation term, respectively. Since the prefactors $1+12p-12p^2$ and $1+6p+36p^2-40p^3$ are positive for $0 \leqslant p \leqslant 1$, the analytical features, which will be given below, are similar to those for $p = 0$. However, the purpose of our analysis is to clarify the effect of taking into account the headway of the immediately preceding car; hence the p-dependence in the analysis is examined in detail.

In order to consider the slowly varying behaviour in the long-wavelength region near the critical point, we introduce a small scaling parameter ε as follows:

$$V'(b_c)\tau = \tfrac{1}{2}(1+2p) + \varepsilon^2. \tag{6}$$

Using equation (5), we introduce the slow variables X and T as follows:

$$X = \varepsilon(n + V'(b_c)t) \tag{7}$$

$$T = \varepsilon^3\frac{1+12p-12p^2}{24}V'(b_c)t \tag{8}$$

and the headway is expressed as

$$\Delta x_n = b_c + \varepsilon\sqrt{\frac{(1+12p-12p^2)V'(b_c)}{4|V'''(b_c)|}}R(X,T). \tag{9}$$

The scale factors in equations (7)–(9) are introduced so as to lead to a simple form of the equation for R, which will be given below. From equation (1), we obtain the equation for the headway as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta x_n(t+\tau) = V(\Delta x_{n+1}) - V(\Delta x_n) + p(V(\Delta x_{n+2}) - 2V(\Delta x_{n+1}) + V(\Delta x_n)). \tag{10}$$

Substituting equations (7)–(9) into (10) and expanding to the fifth order of ε, one finally obtains

$$\partial_T R - \partial_X^3 R + \partial_X R^3 = \varepsilon\frac{24}{1+12p-12p^2}\left\{-\partial_X^2 R - \frac{1}{48}(1+6p+36p^2-40p^3)\partial_X^4 R \right.$$
$$\left. + \frac{1}{48}(1+2p)(1+12p-12p^2)\partial_X^2 R^3\right\} \tag{11}$$

because the terms up to the third order of ε vanish. At the critical point where $\varepsilon = 0$, equation (11) reduces to the MKdV equation. Since we are interested in the steady travelling solutions to the MKdV equation, we consider $R_0(X,T) = R_0(X-cT)$, where the propagation velocity c will be determined from the order of the ε-term in equation (11). Thus one finds the solution to the MKdV equation:

$$R_0 = \sqrt{c}\tanh\left(\sqrt{\frac{c}{2}}(X-cT)\right). \tag{12}$$

Substituting $R = R_0 + \varepsilon R_1$ into equation (11), we get the equation for R_1:

$$LR_1 = M[R_0] \tag{13}$$

where

$$L = c\partial_X + \partial_X^3 - 3(\partial_X R_0^2) - 3R_0^2\partial_X \tag{14}$$

$$M[R_0] = \frac{24}{1+12p-12p^2}\left\{\partial_X^2 R_0 + \frac{1}{48}(1+6p+36p^2-40p^3)\partial_X^4 R_0\right.$$
$$\left. - \frac{1}{48}(1+2p)(1+12p-12p^2)\partial_X^2 R_0^3\right\}. \tag{15}$$

In order to determine the propagation velocity c, we consider the solvability condition for equation (13):

$$(\Phi_0, M[R_0]) \equiv \int_{-\infty}^{\infty}\mathrm{d}X\ \Phi_0 M[R_0] = 0 \tag{16}$$

where Φ_0 is the zeroth eigenfunction of the adjoint operator $L^\dagger$ which is given by

$$L^\dagger\Phi_0 = 0 \qquad L^\dagger = -c\partial_X - \partial_X^3 + 3R_0^2\partial_X. \tag{17}$$

By carrying out the integration in equation (16), we can obtain the value of c:

$$c = \frac{240}{5+54p+108p^2-152p^3}. \tag{18}$$

Using equations (6)–(9), (12), (18), we finally obtain the propagating kink solution

$$\Delta x_n = b_c \pm \sqrt{\frac{30(2V'\tau-1-2p)(1+12p-12p^2)V'}{(5+54p+108p^2-152p^3)|V'''|}}\tanh\left(\sqrt{\frac{60(2V'\tau-1-2p)}{5+54p+108p^2-152p^3}}\right.$$
$$\left.\times\left\{n + V't\times\left(1 - \frac{5(1+12p-12p^2)(2V'\tau-1-2p)}{5+54p+108p^2-152p^3}\right)\right\}\right) \tag{19}$$

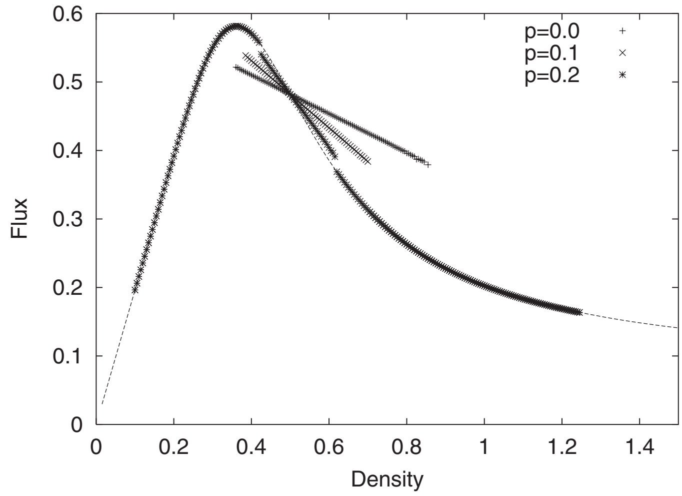

Figure 1. The relation between the flow and the density in the fundamental diagram for $p = 0, 0.1, 0.2$.

where $V' = V'(b_c)$ and $V''' = V'''(b_c)$. From equation (19), one finds that the backward velocity of the congestion increases as the value of p increases. We can find the relation between the headway Δx_n and the delay time τ:

$$\tau = \frac{1+2p}{2} + \frac{5+54p+108p^2-152p^3}{30(1+12p-12p^2)}(\Delta x_n - b_c)^2 \tag{20}$$

which will be compared with the numerical simulation result in the next section.

4. Simulation

We solve the differential–difference equations in equation (1) numerically for various values of p. The initial conditions that we adopt here are

$$x_n(t) = \begin{cases} bn & 0 \leqslant t < \tau \\ bn + y_n(\tau) & t = \tau \end{cases} \tag{21}$$

where $y_n(\tau)$ is taken to be a uniform random distribution between -0.5 and 0.5. Since we are interested in the p-dependence, some fixed value of τ is chosen in the simulation.

One of the important problems for traffic flow is that of investigating the relation between the flux and the density, which is called the fundamental diagram. The density ρ of the cars is defined by N/L, where we choose $L = 200$ and vary N from 20 to 250 in the simulation. The flux Q is defined by the number of cars passing a position per unit time. The data were accumulated and averaged over 20 000 time steps. Numerical results are plotted in figure 1 for $p = 0, 0.1, 0.2$ where we choose $\tau = 0.8$ as an example. In the uniform flow, the relation between flux Q and density ρ is given by

$$Q = \rho V(1/\rho) = \rho\left(\tanh(1/\rho - 2) + \tanh(2)\right). \tag{22}$$

In figure 1, it is represented by the dashed curved line. In the uniform-flow region, the numerical results agree with this curved line. Figure 1 shows that the uniform solution remains stable over a larger range of densities on taking into account the headway of the preceding car.

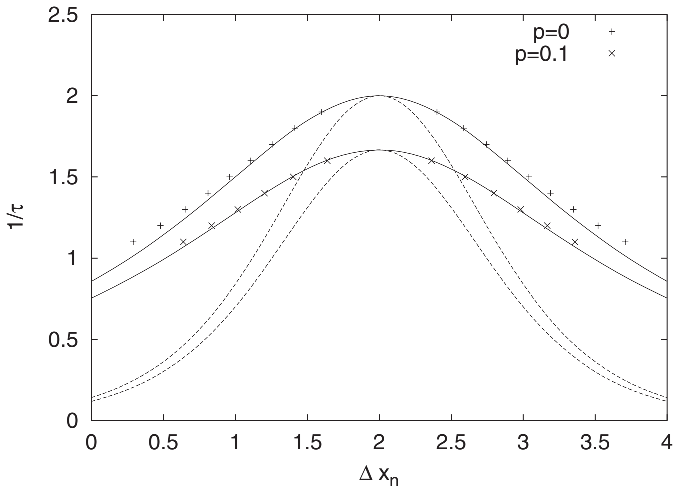

Figure 2. The relation between the headway and $1/\tau$ for the cases of $p = 0$ and 0.1.

We can summarize the effect of the Δx_{n+1}-dependent term in the fundamental diagram as follows: in the uniform-flow region, there is no effect on the flow–density relation, because p-dependence in the generalized optimal velocity function disappears in the case of uniform flow, as is seen in equation (1). In the congested region, the flow increases as p increases if $\rho < 1/2$ and inversely the flow decreases as p increases if $\rho > 1/2$. The reason is as follows. When the density is low, the average of the headways is long. If Δx_{n+1} is long, a larger value of the velocity than that for the case without dependence on Δx_{n+1} is allowed. Hence the flow increases on taking into account the headway of the immediately preceding car. Inversely, when the density is high, i.e. $\rho > 1/2$, a small value of the velocity is taken compared with the case of $p = 0$, because Δx_{n+1} is short. Hence the flow decreases on taking into account the headway of the immediately preceding car.

Finally, we examine the relation between the headway and the delay time for various values of p. In the congested region, stop-and-go states appear. We can obtain the relation between the headway and the inverse of the delay time numerically. Numerical results are plotted in figure 2 for the cases of $p = 0$ and 0.1 as an example. The theoretical curve obtained from equation (20) is drawn as a solid curve. Here the neutral curve obtained from the linear analysis given in equation (4) is also shown, as a dashed curve. We find that the result obtained from the nonlinear analysis is in good agreement with the simulation result near the critical point.

5. Conclusions

We have analysed the generalized optimal velocity model described by a differential–difference equation with a generalized optimal velocity function which depends not only on the headway of each car but also the headway of the immediately preceding one. On taking into account the headway of the immediately preceding car, the model is stabilized for a larger delay time compared with that in the case of $p = 0$. Nonlinear analysis of the model shows that the traffic congestion is described by the MKdV equation at the critical point. The propagating kink solution and the relation between the headway and the delay time are obtained. We found that the analytical result is in good agreement with the numerical simulation near the critical point. In the simulation, the relation between the density and flow has been examined.

We found that the uniform solution remains stable over a larger range of densities on taking into account the headway of the preceding car.

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